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# Topological degeneracy and vortex dynamics in the Kitaev honeycomb model

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We investigate the loop symmetries of Kitaev's honeycomb lattice model. These provide a natural framework to study the abelian topological phase. We show that in the thermodynamic limit, the abelian phase on a torus is topologically degenerate to all orders of the Brillouin-Wigner expansion. We then demonstrate that these symmetries correspond to the evolution of fermions in closed loops. Importantly, we demonstrate that these fermions are made from energetically confined pairs of vortices and that their transport happens with no additional energy cost. This has important implications for the gapped and gapless phases.

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Recently, Kitaev introduced a spin-1/2 quantum lattice model which exhibits abelian and non-abelian topological phases [1]. The system comprises of two-body interactions and is exactly solvable. This simplicity and richness make it attractive both theoretically [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and experimentally [14, 15]. The model is relevant to on-going research into topological quantum information processing [16, 17].

We will analyse the system's closed loop symmetries and show that they correspond to terms in the perturbation series for the abelian phase. We then show that the abelian phase is topologically degenerate to all orders of the perturbation theory around the fully dimerized system. Finally we analyse the quasi-particle excitations created by open string operators. Vortices must be created in pairs and the energy of such a pair increases with separation. However, certain composite vortex-pairs can be separated arbitrarily far without any energy cost. These vortex pairs are fermions and can occur as low-energy excitations. They should not be confused with the fermions introduced as redundant degrees of freedom in [1], those obtained by Jordan-Wigner transformation [4, 5, 8], or with the free-fermionic excitations of [9].

The Hamiltonian for the system can be written as

$$H = - \sum_{\alpha \in \{x,y,z\}} \sum_{i,j} J_{\alpha} K_{ij}^{\alpha,\alpha} \quad (1)$$

where  $K_{ij}^{\alpha,\beta} \equiv \sigma_i^{\alpha} \otimes \sigma_j^{\beta}$  denotes the exchange interaction occurring between the sites  $i, j$  connected by a  $\beta$ -link, see FIG. 1. In what follows we will use  $K_{ij}^{\alpha} \equiv K_{ij}^{\alpha,\alpha}$  whenever  $\alpha = \beta$ . Following [1], we consider loops of  $n$  non-repeating  $K$  operators  $C = K_{ij}^{\alpha(1)} K_{jk}^{\alpha(2)}, \dots, K_{li}^{\alpha(n)}$  where  $\alpha^{(m)} \in x, y, z$ . Any loop constructed in this way commutes with the Hamiltonian and with all other loops. When the model is mapped to free Majorana fermions coupled to a  $\mathbb{Z}_2$  gauge field, these loop operators become Wilson loops [1]. The plaquette operators

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z, \quad (2)$$

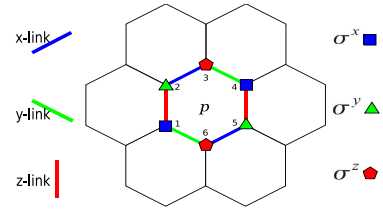


FIG. 1: (color online). The honeycomb lattice and the plaquette operator  $p$ .

where the numbers 1 through 6 in this case label lattice sites on single hexagonal plaquette  $p$ , see FIG. 1, are the closed loop operators around each of the hexagons of the lattice. Since they commute with the Hamiltonian and with each other we may choose energy eigenvectors  $|n\rangle$  such that  $w_p = \langle n | W_p | n \rangle = \pm 1$ . If  $w_p = -1$ , one says that the vector  $|n\rangle$  carries a vortex at  $p$ .

For a finite system of  $N$  spins on a torus there are  $N/2$  plaquettes. The product of all plaquette operators is the identity and this is the single nontrivial relation between them. Hence there are only  $N/2 - 1$  independent quantum numbers,  $\{w_1, \dots, w_{N/2-1}\}$ . All homologically trivial loops (i.e. boundaries) are clearly products of plaquettes. In fact the relevant homology is  $\mathbb{Z}_2$  homology, since loop operators square to the identity. To describe the full symmetry group generated by loop operators, we must introduce generators for the nontrivial  $\mathbb{Z}_2$  homology classes of the surface that the lattice lives on. At most one generator per homology class is necessary, since all elements of any homology class can be generated from an arbitrary element of that class using the plaquettes. The  $\mathbb{Z}_2$  homology group of the torus is  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , so to generate all loop symmetries on the torus, it is enough to add two homologically nontrivial loops as generators (for example the meridian and the longitude of the torus). The third nontrivial class is generated from the product of these two. The full loop symmetry group of the torus is the abelian group with  $N/2 + 1$  independent genera-

tors of order 2, that is  $\mathbb{Z}_2^{N/2+1}$ . The total number of loop symmetries is thus  $2^{N/2+1}$ . All closed loop symmetries can conveniently be written in the form

$$C = G_i F_j (W_1, W_2, \dots, W_{N-1}). \quad (3)$$

Here  $i \in \{0, 1, 2, 3\}$ ,  $G_0 = I$  and  $G_{i>0}$  is any symmetry from each of the three non-trivial homology classes. Of course each non-trivial class can be seen as the product of the other two. The  $F_j$ , with  $j \in \{1, \dots, 2^{N/2-1}\}$ , run through all monomials in the  $W_p$ .

The loop symmetries play an important role in the perturbation theory of the abelian phase of the model. Following Kitaev we take  $J_z \gg J_x, J_y$  and write the Hamiltonian as  $H = H_0 + U$ , where  $H_0 = -J_z \sum K^z$  is the unperturbed Hamiltonian and  $U = -\sum_{\alpha \in \{x, y\}} J_\alpha \sum_{i,j} K^\alpha$  is the perturbative contribution.  $H_0$  has a  $2^{N/2}$  fold degenerate ground state space spanned by configurations in which the spins on  $z$ -links are all dimerized. To investigate the splitting of this degeneracy, one may use the Brillouin-Wigner perturbative expansion [18, 19, 20]. Define  $\mathcal{P}$  to be the projector onto the space of dimerized configurations. For any exact eigenstate of the full Hamiltonian  $|\psi\rangle$  its projection  $|\psi_0\rangle$  onto this subspace satisfies

$$\left[ E_0 + \sum_{n=1}^{\infty} H^{(n)} \right] |\psi_0\rangle = E |\psi_0\rangle = H_{\text{eff}} |\psi_0\rangle, \quad (4)$$

where  $H^{(n)} = \mathcal{P} U \mathcal{G}^{n-1} \mathcal{P}$  and  $\mathcal{G} = [1/(E - H_0)] (1 - \mathcal{P}) U$ . Note that while (4) is an implicit equation for  $E$ , the energy of the state  $|\psi\rangle$ , it is also perturbatively exact.

Calculating the  $n^{\text{th}}$  order correction is equivalent to finding the non-zero elements of the matrix  $H^{(n)}$ . Contributions to  $H^{(n)}$  come from the length  $n$  products  $K_{ij}^{\alpha(1)} \dots K_{lm}^{\alpha(n)}$  with  $\alpha^{(m)} \in x, y$  that preserve the dimerized subspace. Hence any such contribution comes from an element of the group of loop symmetries from which all factors  $K_{ij}^z$  corresponding to  $z$ -links have been removed.

The resulting effective Hamiltonian can be written in terms of operators acting on the spins of the dimers using the following transformation rules:

$$\begin{aligned} \mathcal{P}[\sigma^x \otimes \sigma^y] &\rightarrow +\sigma^y & \mathcal{P}[\sigma^x \otimes \sigma^x] &\rightarrow +\sigma^x \\ \mathcal{P}[\sigma^y \otimes \sigma^y] &\rightarrow -\sigma^x & \mathcal{P}[\sigma^z \otimes I] &\rightarrow +\sigma^z \\ \mathcal{P}[\sigma^z \otimes \sigma^z] &\rightarrow +I \end{aligned} \quad (5)$$

On the left of these rules we have pairs of Pauli operators acting at each end of the dimer, as they would occur in contributions to the effective Hamiltonian (note that all pairs that do not break the dimer are present). On the right we have written the corresponding operators acting on the effective spin of the dimer. A useful property of the transformation from the full spin system to the effective system is that it can be applied directly to loop symmetries without removing the  $z$ -links first, without changing the resulting operator on the dimerized space.

Thus, we need only to examine the effective representations of the projections of the loop symmetries themselves to understand the possible forms of the effective Hamiltonian. The lowest order contributions come from the plaquette operators. We have  $\mathcal{P}[W_p] \rightarrow Q_p = \sigma_l^y \sigma_r^y \sigma_u^z \sigma_d^z$ , where  $l, r, u, d$  denotes the relative positions (left, right, up and down) of the effective spins. Expanding to all orders, we have contributions from all loop symmetries, both topologically trivial and non-trivial. To come to an explicit expression for the effective Hamiltonian, we now introduce a particular generating set for the loop symmetry group, constructed from  $N/2 - 1$  plaquettes and the operators  $Z$  and  $V$  defined as  $Z \equiv \prod_i \sigma_i^z$ , where  $i$  represents lattice sites in the horizontal direction of alternating  $x$  and  $y$ -links and  $V \equiv \prod K_{j,k}^{x,y} \prod K_{l,m}^{y,x}$ , where the products take place over vertically arranged links. The projections  $\mathcal{P}(Z)$  and  $\mathcal{P}(V)$  correspond to operators  $z$  and  $y$  on the dimerized space, which act by  $\sigma^z$  and  $\sigma^y$  on the relevant effective spins, (see FIG. 2). In analogy to (3) we can now write the full effective Hamiltonian as

$$H_{\text{eff}} = \sum_{i=0}^3 \sum_{j=1}^{2^{N/2-2}} d_{i,j} G_i(z, y) F_j(Q_1, Q_2, \dots, Q_{N/2-2}), \quad (6)$$

where  $G_0 = I, G_1 = z, G_2 = y$  and  $G_3 = zy$  and the  $d_{i,j}$  are constants which depend on  $J_x, J_y$  and  $J_z$ . This form is strictly valid for when effective square toroidal lattice has even number of plaquettes  $Q_p$  along both directions. The arguments given here can be easily generalised to odd-odd and odd-even lattices examined in [21].

The homologically non-trivial contributions are often called ‘finite size effects’. Their coefficients  $d_{i,j}$  with ( $i > 0$ ) are of the order  $O(J_x^{n_x} J_y^{n_y})$ , where  $n_x$  and  $n_y$  are the respective number of  $x$ -links and  $y$ -links used to make  $G_i(z, y) F_j(Q_1, Q_2, \dots, Q_{N/2-2})$ . In the thermodynamic limit these terms vanish and the effective Hamiltonian is dominated by contributions of the form  $F_j(Q_1, Q_2, \dots, Q_{N/2-2})$ . Note that this form is similar to the thermodynamic planar Hamiltonian addressed in [9].

We can now address the topological degeneracy of the abelian phase. The general argument for topological ground state degeneracy depends on the existence of operators  $T_1$  and  $T_2$  that create particle/anti-particle pairs from the vacuum, bring the particle around the torus and then annihilate both particles [22, 23, 24]. These operators should commute with the Hamiltonian but, crucially, not with each other. It is therefore clear that  $T_1$  and  $T_2$  operators for the honeycomb system cannot be contained within the group of commuting loop symmetries. However, the low-energy effective representations of the homologically nontrivial loops’ generators have the factorizations  $z = z_b z_w$  and  $y = y_b y_w$ , where  $z_b$  and  $y_b$  act with effective  $\sigma^z$ ’s and  $\sigma^y$ ’s respectively on the spins of the ‘black’ dimers involved in  $z$  and  $y$ , while  $z_w$  and  $y_w$  do the same for the ‘white’ dimers, (see FIG. 2).

These black and white operators correspond precisely

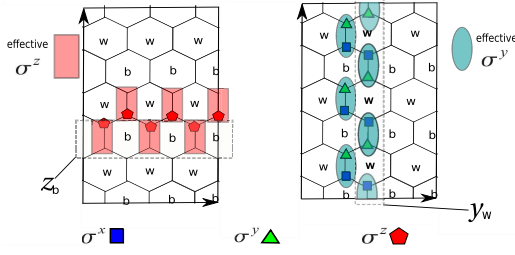


FIG. 2: (color online). The  $Z$  and  $V$  chains with their projections on to the dimerized subspace. The projections may be factorised into products  $\mathcal{P}[Z] \rightarrow z_b z_w$  and  $\mathcal{P}[V] \rightarrow y_b y_w$ . Each of the individual factors  $z_b, z_w, y_b, y_w$  also commute with the effective Hamiltonian but obey the relation  $z_j^{-1} y_k^{-1} z_j y_k = e^{i\pi(1-\delta_{jk})} I$ .

to the nontrivial loop operators on the square lattice and dual square lattice of the toric code (cf. [22]) and thus obey the commutation relations  $z_j^{-1} y_k^{-1} z_j y_k = e^{i\pi(1-\delta_{jk})} I$ . These operators commute with the effective plaquette operators  $Q_p$ , and hence with the 4<sup>th</sup> order effective Hamiltonian, which is a toric code Hamiltonian. This means that, at least to 4<sup>th</sup> order, one may take the operators  $z_b, z_w, y_b$  and  $y_w$  as approximations for the  $T_1$  and  $T_2$  symmetries for the full effective Hamiltonian  $H_{\text{eff}}$ . However, since these operators commute with plaquette operators, they must also commute with all homologically trivial components of  $H_{\text{eff}}$ .

Hence we see that for large tori, the topological degeneracy of the system is much deeper than the 4<sup>th</sup> order approximation alone. In fact, in the thermodynamic limit it is valid to all orders of the perturbation theory. The eigenstates of the effective Hamiltonian are still those of the toric code and indeed, like the toric code, each quasi-particle configuration is 4-fold degenerate. However, unlike the toric code, the energies of different configurations of the same quasi-particle number can be different [9].

We now concentrate on the full Hamiltonian and consider what the physical properties associated with open ended strings of the  $K$  and  $\sigma$  operators. We first note that  $\{\sigma_i^\alpha, W_p\} = 0$  when the site  $i$  belongs to an  $\alpha$ -link at plaquette  $p$ . Hence, the operator  $\sigma_j^\alpha$  changes the vorticity of the two plaquettes sharing this  $\alpha$ -link by either creating or annihilating a pair of vortices, or moving a vortex from one plaquette to the other. In terms of the  $K$  operators we may write  $[K_{ij}^\alpha, W_p] = 0$  ( $\forall i, j$ ),  $[K_{ij}^{\alpha,\beta}, W_p] = 0$  ( $i, j \notin p$ ) and  $\{K_{ij}^{\alpha,\beta}, W_p\} = 0$  ( $i, j \in p$ ). Now note that

$$K_{ij}^\alpha H K_{ij}^\alpha = H + 2J_\beta(K_{ik}^\beta + K_{jl}^\beta) + 2J_\gamma(K_{im}^\gamma + K_{jn}^\gamma), \quad (7)$$

$$K_{ij}^{\alpha,\beta} H K_{ij}^{\alpha,\beta} = H + 2J_\gamma(K_{ik}^\gamma + K_{jl}^\gamma), \quad (8)$$

where  $\alpha \neq \beta \neq \gamma$  and  $k, l, m, n$  are neighboring sites of  $i$  and  $j$  along  $\beta$ - or  $\gamma$ -links. Energy eigenstates must obey

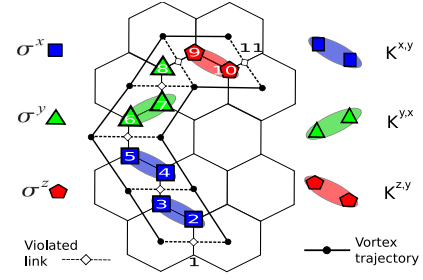


FIG. 3: (color online). The operator  $K_{2,3}^{x,y}$  is used to create two vortex-pairs from the vacuum with an energy cost of  $2J_z\langle K_{1,2}^z + K_{3,4}^z \rangle$ . The subsequent operators  $K_{4,5}^{x,y}$  and  $K_{6,7}^{y,x}$  move one of the pairs in the direction shown. The Pauli operator  $\sigma_8^y$  rotates this pair and the energy of the system is at this point is  $E_0 + 2J_z\langle K_{1,2}^z \rangle + 2J_x\langle K_{8,9}^x \rangle$ . This new pair is then moved horizontally with no additional energy cost by  $K_{9,10}^{z,y}$ .

$\langle K_{ij}^\alpha H K_{ij}^\alpha \rangle = E_n + 2J_\beta\langle K_{ik}^\beta + K_{jl}^\beta \rangle + 2J_\gamma\langle K_{im}^\gamma + K_{jn}^\gamma \rangle$  and  $\langle K_{ij}^{\alpha,\beta} H K_{ij}^{\alpha,\beta} \rangle = E_n + 2J_\gamma\langle K_{ik}^\gamma + K_{jl}^\gamma \rangle$ . If these expectation values are taken with respect to the ground state  $|0\rangle$ , which is from the vortex-free sector [1, 25], and  $\langle K_{ij}^\alpha \rangle \neq 0$  and  $\langle K_{ij}^{\alpha,\beta} \rangle \neq 0$  hold, then the states  $K_{ij}^\alpha|0\rangle$  and  $K_{ij}^{\alpha,\beta}|0\rangle$  should have energies  $E \geq E_0$ . Thus, in general, both  $K_{ij}^\alpha$  and  $K_{ij}^{\alpha,\beta}$  increase the energy of the system, but due to the different commutation relations with the plaquette operators, only the latter changes the vorticity on the four plaquettes adjacent to the link  $(ij)$ .

Define a path  $s$  on the lattice as some ordered set of  $|s|$  neighboring sites connecting the endpoints  $i$  and  $j$ . A string operator along this path,  $C_{ij}^s(K, \sigma)$ , can be defined as a site ordered product of  $\sigma^\alpha$  and  $K^{\alpha,\beta}$  operators acting on sites of  $s$ . For the moment let us assume that the operators in  $C$  act on sites such that there is no overlap between any of the  $K$ 's or  $\sigma$ 's. Such an operator first creates two vortex-pairs and subsequently moves one of the pairs along the path  $s$ . We see that  $\sigma^\alpha$  correspond to a rotation of the vortex-pair, whereas  $K^{\alpha,\beta}$  moves it without a rotation (see FIG. 3). If  $i$  and  $j$  are neighboring sites and  $s$  is a homologically trivial loop then by definition  $C_{(ij)}^s = \prod_p W_p$ , where the product is over all plaquettes enclosed by  $s$ . If we treat a vortex-pair as a composite particle then the simplest loop operator  $C_{(ij)}^p = W_p$  has two interpretations: (I) the composite particle is rotated by  $2\pi$  or, (II) one vortex is transported around the other. The resulting overall phase of  $e^{i\pi}$  implies that the composite particles are fermions whereas the vortices are anyons [26].

Since closed strings  $C_{(ij)}^s$  are symmetries of the Hamiltonian, their action does not increase the energy. On the other hand, open string operators satisfy

$$\langle C_{ij}^s H C_{ij}^s \rangle = E_0 + 2J_\gamma\langle K_{ik}^\gamma \rangle + 2J_\tau\langle K_{jl}^\tau \rangle, \quad \forall s \quad (9)$$

where the first and last links along the path  $s$  are  $\alpha$ - and

$\mu$ -links, respectively, and the operators acting on them are  $\sigma^\beta$  or  $K^{\alpha,\beta}$  and  $\sigma^\nu$  or  $K^{\nu,\mu}$ , respectively, (see FIG. 3). The sites  $k$  and  $l$  (1 and 11) are the nearest neighbors to  $i$  and  $j$  (2 and 10) along  $\gamma$ - and  $\tau$ -links (z- and x-links), respectively. The crucial point is that the number of extra terms on the right hand side of (9) is insensitive to  $s$ . Deforming the path by  $s \rightarrow s'$  may change  $\gamma$  and  $\tau$ , but the number of terms stays constant. Since these extra terms are responsible for increasing the energy of the system, the insensitivity to  $s$  implies that the vortex-pair can be propagated freely up to local potentials created by non-homogeneous configurations of the couplings  $J_\alpha$ . There are two special cases. First, at the special point  $J_x = J_y = J_z$  these vortex-pairs are free to move arbitrary paths without increasing the energy of the system. Second, regardless of the coupling configuration, the vortex-pairs can be moved without costing energy if their relative orientation is maintained, i.e. when  $C_{ij}^s$  does not involve any  $\sigma^\alpha$  operators.

This is in contrast to the separation of vortices which, can only be done by applying broken string operators,  $D_{ij}^s$ , i.e. loop operators with certain  $\sigma^\alpha$ 's removed. In general such operators satisfy

$$\langle D_{ij}^s H D_{ij}^s \rangle = E_0 + a J_x \langle K^x \rangle + b J_y \langle K^y \rangle + c J_z \langle K^z \rangle, (10)$$

where  $a + b + c = |s|$  for some integers  $a$ ,  $b$  and  $c$  depending on the path  $s$ . The number of terms on right hand side scales with the length  $|s|$  of the string, which implies that if these terms give an energy penalty, then there is string tension. We have confirmed (10) numerically using ground states obtained by exact diagonalization of a 16-spin toroidal system. Details of these calculations will be presented elsewhere [27]. At small but finite temperatures the vortices are thus confined and they pair up to form fermionic vortex-pairs [28].

The observation of the string tension and the presence of low-energy vortex-pairs provides another perspective to understand the lack of topological phase when  $J_x \approx J_y \approx J_z$  [1]. Moving vortices increases the energy of the system, and thus it is not possible to control whether vortex-pairs are spontaneously also excited. Since the pairs and the vortices have mutual anyonic statistics, the potential presence of unaccounted vortex-pairs thus make it impossible to define the relevant statistics.

In conclusion, we have associated each of the  $2^{N/2+1}$  loop symmetries of the full system with a particular perturbative term in the Brillouin-Wigner expansion. We then demonstrated that, in the thermodynamics limit, the topological degeneracy of the system remains to all orders of the perturbation theory. We then interpreted the loop operators acting on the full system ground state, as the creation of a pair of fermions, propagation along a closed path and subsequent annihilation back to vacuum. By decomposing these operations into particular products we showed how these particular fermions can

be propagated around the lattice with no additional energy cost. The vortices which make up each fermion are by themselves abelian anyons. We show that to move individual vortices apart costs energy that scales with the length of a connecting broken operator string.

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